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A NEW ANALYTICAL EXPRESSION FOR THE DRAG OF A FLAT PLATE VALID FOR BOTH THE TURBULENT AND LAMINAR REGIMES

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Abstract—An analytical expression for the "law of the wall", valid for the laminar, turbulent and intermediate regions, is shown to permit the derivation of explicit relations connecting: the local drag coefficient and the momentum-thickness Reynolds Number; the local drag coefficient and the length Reynolds Number; and the overall drag coefficient and the length Reynolds Number. Agreement is good with the Prandtl–Schlichting formula at high Reynolds Numbers and the Blasius solution at low Reynolds numbers. An extension to flows with stream-wise pressure gradients is presented.

NOMENCLATURE

- c_f, local drag coefficient, row 7 of Table 1;
- $\bar{c}_f$, overall drag coefficient, row 8 of Table 1;
- E, a constant, taken as 9·025 in the present report though 12·0 gives a better fit;
- H_{12}, shape factor, row 9 of Table 1;
- K, a constant, taken as 0·4 in the present report;
- N_{Pr}, Prandtl number;
- N_{St}, Stanton number;
- R, $\int_0^x (u_G/\nu)\, dx$;
- u, local time mean velocity along the surface;
- u_G, value of u in main stream;
- u^+, $u/(\tau_S/\rho)^{1/2}$;
- w, function of u_G^+ defined by equation (11);
- x, distance along the surface in stream direction;
- x^+, non-dimensional distance defined by equation (2);
- y, distance from wall;
- y^+, non-dimensional distance defined in row 1 of Table 1;
- z, function of u_G^+ defined in row 5 of Table 1;
- δ_1, displacement thickness, $\int_0^{y_G} (1 - u/u_G)\, dy$;
- δ_1^+, non-dimensional displacement thickness, row 3 of Table 1;
- δ_2, momentum thickness, $\int_0^{y_G} (u/u_G)(1 - u/u_G)\, dy$;
- δ_2^+, non-dimensional momentum thickness, row 4 of Table 1;
- ϵ^+, non-dimensional total viscosity ($\nu_{total}/\nu_{laminar}$), row 2 of Table 1;
- ν, kinematic viscosity of fluid;
- τ_S, shear stress at wall;
- ρ, density of fluid.

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1. PURPOSE OF THE PRESENT NOTE

KESTIN and PERSEN [1] have made an ingenious use of the turbulent-boundary-layer heat-transfer theory of the present author [2] to derive an expression for the drag exerted by a turbulent boundary layer on a flat plate. They compare the drag so derived with experimental data, suggesting that the comparison affords a test of the heat-transfer theory; the predicted drag is found to exceed the experimental drag by approximately 10 per cent. The purpose of the present note is to examine the source of this discrepancy so as to establish whether it is due to an inadequacy of the universal velocity profile on which both theories are based, or to the assumptions necessarily made in the Kestin–Persen extension.

In the course of this examination it will be shown that the velocity profile can be used, without further assumptions, to derive an analytical expression for the drag law which agrees well with the experimental data. A simple modification enables the data for both laminar and turbulent boundary layers to be fitted by the same expression: no such fit has been achieved in the transition region however. Finally, the theory will be extended to the calculation of drag on two-dimensional bodies for which the free-stream velocity varies with distance.

2. THE KESTIN–PERSEN THEORY

The heat-transfer theory of the present author [2] rests on two main assumptions, namely:

(i) all velocity profiles in a developing boundary layer can be represented by a single "universal" function $y^+(u^+)$ in the usual notation: they are terminated by a "kink" at the outer edge of the boundary layer, such that: for $y > y_G$, $u = u_G$. An analytical expression for the function has been developed.

(ii) The thermal boundary layer is considerably thinner than the velocity boundary layer; this holds, for example, when the point on the wall from which the thermal layer springs is far downstream of the point of origin of the velocity boundary layer. The assumption may be expressed as a relation between Stanton number, Prandtl number and drag coefficient as follows:

$$N_{St}\, N_{Pr} \gg c_f/2. \qquad (1)$$

The theory has led to the derivation of the quantity $N_{St}/(c_f/2)^{1/2}$ as a function of N_{Pr} and x^+, the latter being a non-dimensional distance downstream of the point at which the thermal layer starts, defined by:

$$x^+ \equiv \int_0^x (c_f/2)^{1/2}(u/\nu)\, dx. \qquad (2)$$

Kestin and Persen [1] have evaluated the function numerically for a Prandtl number of unity.

In order to use the above function as a means to the computation of the drag of a flat plate, Kestin and Persen made an additional assumption.

(iii) This was that the aforesaid functional relationship remains valid even when condition (i) is contravened, as it must be when the thermal and velocity layers spring from the same point: for, in this case, and with $N_{Pr} = 1$, the Reynolds analogy applies, viz.

$$N_{St} = c_f/2. \qquad (3)$$

Mathematical manipulation then yielded the required expression for c_f in terms of length Reynolds number.

As has been mentioned already, the values of c_f so obtained differ systematically from experimental values. It is therefore necessary to decide whether these differences are due to the inconsistency between (ii) and (iii) or to errors in (i). The latter assumption could be wrong altogether; or the form assumed for $y^+(u^+)$ could be erroneous. We shall therefore now derive a drag law using assumption (i) alone, and shall compare it with experimental data.

3. ANALYTICAL EXPRESSIONS FOR BOUNDARY-LAYER THICKNESSES, REYNOLDS NUMBERS AND DRAG COEFFICIENTS

When the velocity distribution is given, the momentum thickness δ_2 can be evaluated by means of a quadrature.

When the velocity distribution is expressed in terms of universal co-ordinates, the quadrature yields a "local" drag law, i.e. a connection between the local drag coefficient c_f and the local Reynolds Number based on momentum thickness, $u_G\, \delta_2/\nu$, where u_G is the velocity of the main stream and ν is the kinematic viscosity.

If u_G is independent of distance x from the leading edge, as is true for the flat plate, insertion of the local drag law into the von Kármán momentum-integral equation, followed by integration, leads to a relation between c_f and the Reynolds number based on length, $u_G\, x/\nu$. A further integration yields values of $\bar{c}_f$, the overall drag coefficient.

Now the author's heat-transfer theory [2] is based on the assumption that the relation between the non-dimensional distance y^+ and the non-dimensional velocity u^+ has the form represented by the first line of Table 1 [3], where K and E are constants. This relation, as well as being in rather satisfactory agreement with the

Table 1

Row no.	Name of quantity (non-dimensional)	Symbol	Definition	Analytical expression	Asymptotic expression as $u^+ \to 0$	Asymptotic expression as $u^+ \to \infty$
1.	Distance from wall	y^+	$y\sqrt{(\tau_S/\rho)}/\nu$	$u^+ + (1/E)[e^{Ku^+} - 1 - (Ku^+) - (Ku^+)^2/2! - (Ku^+)^3/3! - (Ku^+)^4/4!]$	u^+	$(1/E)e^{Ku^+}$
2.	Total viscosity*	ϵ^+	dy^+/du^+	$1 + (K/E)[e^{Ku^+} - 1 - (Ku^+) - (Ku^+)^2/2! - (Ku^+)^3/3!]$	1	$(K/E)e^{Ku^+}$
3.	Displacement thickness†	δ_1^+	$\int_0^{y_G^+} [1 - (u/u_G)]\, dy^+$	$u_G^+/2 + (1/E)[e^{Ku_G^+}/Ku_G^+ - 1/Ku_G^+ - 1 - Ku_G^+/2! - (Ku_G^+)^2/3! - (Ku_G^+)^3/4! - (Ku_G^+)^4/5!]$	as $u_G^+ \to 0$ $u_G^+/2$	as $u_G^+ \to \infty$ $\dfrac{e^{Ku_G^+}}{E\,Ku_G^+}$
4.	Momentum thickness†	δ_2^+	$\int_0^{y_G^+} (u/u_G)[1 - (u/u_G)]\, dy^+$	$u_G^+/6 + (1/E)\{e^{Ku_G^+}[(1/Ku_G^+) - 2/(Ku_G^+)^2] + 2/(Ku_G^+)^2 + 1/Ku_G^+ - Ku_G^+/6 - (Ku_G^+)^2/12 - (Ku_G^+)^3/40 - (Ku_G^+)^4/180\}$	$u_G^+/6$	$\dfrac{e^{Ku_G^+}}{E\,Ku_G^+}\left(1 - \dfrac{2}{Ku_G^+}\right)$
5.	Reynolds number‡ based on x†	$u_G\, x/\nu$†	$\int_0^{u_G \delta_2/\nu} (2/c_f)\, d(u_G\, \delta_2/\nu)$ $(\equiv z)$	$(u_G^+)^4/12 + (1/K^3E)\{e^{Ku_G^+}[6 - 4\,Ku_G^+ + (Ku_G^+)^2] - 6 - 2Ku_G^+ - (Ku_G^+)^4/12 - (Ku_G^+)^5/20 - (Ku_G^+)^6/60 - (Ku_G^+)^7/252\}$	$(u_G^+)^4/12$	$\dfrac{(u_G^+)^2}{KE}\, e^{Ku_G^+} \cdot \left[1 - \dfrac{4}{Ku_G^+} + \dfrac{6}{(Ku_G^+)^2}\right]$
6.	Reynolds number based on δ_2†	$u_G\, \delta_2/\nu$	$u_G^+\, \delta_2^+$	$(u_G^+)^2/6 + (1/KE)[e^{Ku_G^+}(1 - 2/Ku_G^+) + 2/Ku_G^+ + 1 - (Ku_G^+)^2/6 - (Ku_G^+)^3/12 - (Ku_G^+)^4/40 - (Ku_G^+)^5/180]$	$(u_G^+)^2/6$	$\dfrac{e^{Ku_G^+}}{EK}\left(1 - \dfrac{2}{Ku_G^+}\right)$
7.	Local drag coefficient	c_f	$\tau_S/(\frac{1}{2}\rho\, u_G^2)$	$2/(u_G^+)^2$	$2/(u_G^+)^2$	$2/(u_G^+)^2$
8.	Overall drag coefficient§	$\bar{c}_f$	$(1/x)\int_0^x c_f\, dx$	$2(u_G\, \delta_2/\nu)/(u_G\, x/\nu)$	$4/(u_G^+)^2$	$[2/(u_G^+)^2] \times \dfrac{1 - 2/Ku_G^+}{1 - 4/Ku_G^+ + 6/(Ku_G^+)^2}$
9.	Shape factor	H_{12}	δ_1/δ_2	δ_1^+/δ_2^+	3	$1/(1 - 2/Ku_G^+)$

* J. Kestin (private communication) has pointed out that to refer to ϵ^+ as the total viscosity is misleading; it is really equal to τ/τ_S times the dimensionless total viscosity. τ/τ_S is less than unity in the outer regions of the boundary layer.

† As explained in Section 4 of the text, more accurate expressions are obtained by replacing the denominator of the first terms of the analytical expressions in rows 3, 4, 5 and 6 by 1·7508, 4·5362, 9·0724 and 4·5362 respectively.

‡ Valid only for the flat plate. When u_G varies with x, $u_G\, x/\nu$ should be replaced by the right-hand side of equation (13).

§ The whole of this line is valid only for the flat plate (u_G independent of x).

that the quantity on the right-hand side of (13) or (14) is put in the place of the length Reynolds number.

Of course this theory is still based on the assumed existence of a universal velocity profile at all stations along the surface. Its validity is therefore restricted to boundary layers for which the pressure gradient is nowhere so steep as to cause changes in velocity profile of the kind which, in extreme cases, leads to boundary-layer separation.

ACKNOWLEDGEMENTS

The author's thanks are due to Miss M. P. Steele for the calculations embodied in Table 2, and to Professor J. Kestin for valuable comments.

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Résumé—Une expression analytique de la condition de paroi valable pour les régions laminaire, turbulente et de transition permet d'expliciter des relations entre le coefficient de traînée local et le nombre de Reynolds lié à l'épaisseur de quantité de mouvement; le coefficient de traînée local et le nombre de Reynolds lié à la distance; et le coefficient de traînée global et le nombre de Reynolds relatif à la distance. L'accord est bon avec la formule de Prandtl–Schlichting pour les nombres de Reynolds élevés et la solution de Blasius pour les bas nombres de Reynolds. L'étude est étendue aux écoulements avec gradient de pression.

Zusammenfassung—Es wird gezeigt, dass für das "Wandgesetz" ein analytischer Ausdruck für den Laminar-, den Turbulenz- und den Zwischenbereich gültig ist, der die Ableitung expliziter Beziehungen erlaubt zwischen: dem örtlichen Schleppwiderstandsbeiwert und der mit der hydrodynamischen Grenzschichtdicke gebildeten Reynoldszahl; dem örtlichen Schleppwiderstandsbeiwert und der mit der Länge gebildeten Reynoldszahl; dem Gesamtschleppwiderstandsbeiwert und der mit der Länge gebildeten Reynoldszahl. Bei grossen Reynoldszahlen ist die Übereinstimmung gut mit der Prandtl-Schlichting-Formel, bei kleinen Reynoldszahlen mit der Blasiuslösung. Eine Erweiterung auf Strömungen mit Druckabfall in Stromrichtung ist angegeben.

Аннотация—Показано, что аналитическое выражение для пластины, применимое для турбулентного, ламинарного и промежуточного режимов, позволяет вывести простые соотношения, связывающие: локальный коэффициент сопротивления и число Рейнольдса для толщины потери импульса; локальный коэффициент сопротивления и число Рейнольдса для длины; общий коэффициент сопротивления и число Рейнольдса для длины. Результаты хорошо согласуются с формулой Прандтля-Шлихтинга при больших значениях чисел Рейнольдса и с решением Блазиуса при их малых значениях. Показано применение выражения для течений с продольным градиентом давлений.

Tvble 2. Quantities evaluated from the formulae of Table 1, with $K = 0{\cdot}4$, $E = 9{\cdot}025$

u_G^+	δ_1^+	δ_2^+	H_{12}	$u_G\,\delta_2/\nu$	$u_G\,x/\nu$	$c_f/2$	$\bar{c}_f/2$	$\dfrac{\bar{c}_f(\log_{10} u_G\,x/\nu)^{2\cdot58}}{0\cdot455}$
3·5	1·7510	0·5841	2·9979	2·0443	1·2529(1)	0·08163	0·1632	0·9129
4·0	2·0021	0·6682	2·9963	2·6727	2·1406(1)	0·0625	0·1249	1·1470
4·5	2·2539	0·7528	2·9940	3·3876	3·4365(1)	0·04938	0·09858	1·3116
5·0	2·5068	0·8383	2·9905	4·1913	5·2551(1)	0·04	0·07976	1·4218
6·0	3·0182	1·0132	2·9787	6·0794	1·1018(2)	0·02777	0·05518	1·5304
7·5	3·8123	1·2957	2·9423	9·7175	2·7844(2)	0·01777	0·03490	1·5398
9·0	4·6756	1·6298	2·8689	1·4668(1)	6·1960(2)	0·01235	0·2367	1·4714
10·0	5·3250	1·9079	2·7910	1·9079(1)	1·0199(3)	0·01	0·01871	1·4101
12·0	6·9789	2·7344	2·5523	3·2813(1)	2·7176(3)	0·006944	0·01207	1·2798
14·0	9·6118	4·3151	2·2275	6·0411(1)	7·4804(3)	0·005102	0·008076	1·1685
16·0	1·4421(1)	7·5975	1·8981	1·2156(2)	2·1509(4)	0·003906	0·005652	1·0915
18·0	2·3925(1)	1·4607(1)	1·6380	2·6292(2)	6·3080(4)	0·003086	0·004168	1·0484
20·0	4·3391(1)	2·9527(1)	1·4696	5·9053(2)	1·8401(5)	0·0025	0·003209	1·0246
22·0	8·3807(1)	6·1773(1)	1·3567	1·3590(3)	5·2663(5)	0·002066	0·002581	1·0214
24·0	1·6813(2)	1·3022(2)	1·2912	3·1252(3)	1·4727(6)	0·001736	0·002122	1·0194
26·0	3·4439(2)	2·7560(2)	1·2496	7·1655(3)	4·0267(6)	0·001479	0·001779	1·0197
28·0	7·1340(2)	5·8400(2)	1·2215	1·6352(4)	1·0794(7)	0·001276	0·001515	1·0211
30·0	1·4872(3)	1·2381(3)	1·2012	3·7143(4)	2·8451(7)	0·001111	0·001306	1·0227
32·0	3·1131(3)	2·6268(3)	1·1851	8·4057(4)	7·3938(7)	0·000977	0·001137	1·0238
34·0	6·5371(3)	5·5774(3)	1·1721	1·8963(5)	1·8988(8)	0·000865	0·000999	1·0254
36·0	1·3764(4)	1·1856(4)	1·1609	4·2682(5)	4·8273(8)	0·000772	0·000884	1·0264
38·0	2·9054(4)	2·5237(4)	1·1512	9·5901(5)	1·2167(9)	0·000693	0·000788	1·0281
40·0	6·1471(4)	5·3798(4)	1·1427	2·1519(6)	3·0438(9)	0·000625	0·000707	1·0304

(i) The numbers in brackets indicate the power of 10 by which the preceding number should be multiplied.
(ii) When u_G is not uniform, $u_G\,x/\nu$ should be replaced by z, as explained in Section 5 of this report. The $\bar{c}_f/2$ column should not then be used.

amended Table 1 expressions may be accepted as giving rather accurately the highest possible drag, which will be achieved in practice whenever tripping devices are present to cause transition to turbulence at the lowest possible $u_G\,x/\nu$.

5. EXTENSION TO FLOWS WITH PRESSURE GRADIENTS

The opportunity will also be taken to apply the theory to flows in which the stream velocity is not uniform. The momentum integral equation for a two-dimensional flow is:

$$\frac{d\delta_2}{dx} + (H_{12}+2)\frac{\delta_2}{\delta_G}\frac{du_G}{dx} = \frac{c_f}{2}. \tag{9}$$

On rearrangement so as to have a function of u_G^+ as the dependent variable, this becomes:

$$\frac{dz}{dR} + z\,.\,w\,.\,\frac{1}{u_G}\frac{du_G}{dR} = 1 \tag{10}$$

where $z(u_G^+)$ is defined in column 5 of Table 2

$$w(u_G^+) \equiv \frac{(u_G^+)^2[(u_G\,\delta_2/\nu)+(u_G\,\delta_1/\nu)]}{z} \rightarrow \frac{2\,.\,(1-1/Ku_G^+)}{1-4/Ku_G^+ + 6/(Ku_G^+)^2} \tag{11}$$

as u_G^+ becomes large,

$$R \equiv \textstyle\int_0^x (u_G/\nu)\,dx \tag{12}$$

and assumption (i) of Section 2 has been invoked. The solution of (10) may be written formally as:

$$z = \exp\{-\textstyle\int (w/u_G)\,du_G\} \times \textstyle\int_0^R \exp\{\int (w/u_G)\,du_G\}\,dR \tag{13}$$

Since c_f is related to z via rows 5 and 7 of Table 1, this equation is the solution to our problem. Actually, w is not quite independent of u_G^+ or z, so iteration is required; however the dependence is very weak, leading to rapid convergence. It is particularly convenient to take w as a constant; then (13) reduces to:

$$z = u_G^{-w}\textstyle\int_0^x (u_G^{w+1}/\nu)\,dx \tag{14}$$

If the Reynolds number is high, (10) shows that w tends to the value 2; a better approximation is:

$$w \approx 2(1 + 3/Ku_G^+) \tag{15}$$

so that, with $u_G^+ = 15$ for example, $w \approx 3$.

The method of applying the solution to the calculation of the local drag coefficient when u_G varies with x can now be expressed as follows: c_f may be obtained from Table 2 provided that the quantity on the right-hand side of (13) or (14) is put in the place of the length Reynolds number.

Of course this theory is still based on the assumed existence of a universal velocity profile at all stations along the surface. Its validity is therefore restricted to boundary layers for which the pressure gradient is nowhere so steep as to cause changes in velocity profile of the kind which, in extreme cases, leads to boundary-layer separation.

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